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Нека V и V' са ЛП и φ е изображение от $V \to V'$

$\varphi : V \to V'$

Изображението φ наричаме линейно ако образът на дадена линейна комбинация от V е същата линейна комбинация във V' ако $a_1 - a_m \in V$ и $\lambda_1 - \lambda_m \in F$

$$\varphi : (a_1\lambda_1 + \dots a_m\lambda_m) = \lambda_1 \varphi(a_1) + \dots + \lambda_m \varphi(a_m)$$

$Hom(VV')$ - множеството от изображения от $V \to V'$

Ако $V \equiv V'$, линейното изображение φ наричаме линеен оператор

$Hom V$ - множеството от всички линейни оператори

Св-ва

1) $\varphi : V \to V'$ е линейно $\Leftrightarrow$

а) $\varphi(a+b) = \varphi(a) + \varphi(b)$, $\forall a\, b \in V$

б) $\varphi(\lambda a) = \lambda \varphi(a)$ $\quad \forall \lambda \in F$, $\forall a \in V$

$\varphi(\lambda a + \mu b) = \lambda \varphi(a) + \mu \varphi(b)$

2) $\varphi(O) = O'$ $\quad (O \in V, O' \in V')$ (нулево множество)

3) $\varphi(-a) = -\varphi(a)$ $\quad \forall a \in V$

4) Ако $a_1 - a_k$ са ЛЗ във V тогава и $\varphi(a_1) - \varphi(a_k)$ са ЛЗ във V'

Нека $O : V \to V'$ е нулевото изображение т.е. $\varphi(a) = O'$ (нулевият вектор на V')

$id : V \to V'$ е идентитет на V т.ще $\forall a \in V \Rightarrow$

$$id(a) = a$$

Примери

1) $\mathcal{O}$ е линейно изображение

2) id е линейно изображение

3) $\delta : F_{[x]} \to F_{[x]}$ което $f \in F_{[x]}$ и $\delta(f) = f(x)$

4) Нека $V = M_n(F)$, $\varphi : V \to V$, $\varphi(A) = A + A^t$ за $\forall A \in M_n(F)$

Ако a и $b \in V$ и $\lambda, \mu \in F$ тогава

1) $\varphi(\lambda a + \mu b) = \mathcal{O}' = \lambda\varphi(a) + \mu\varphi(b)$

2) $id(\lambda a + \mu b) = \lambda id(a) + \mu id(b)$

3) Нека $f(x) \in F_{[x]}$, $g(x) \in F_{[x]}$, $\delta(f) = g'$, $\lambda, \mu \in F$

$$\delta(\lambda f(x) + \mu g(x)) = \lambda\delta(f(x)) + \mu\delta(g(x))$$

4) $A, B \in M_n(F)$ $\lambda, \mu \in F$ тогава

$$\varphi(\lambda A + \mu B) = \lambda A + \mu B + (\lambda A + \mu B)^T = \lambda(A + A^T) + \mu(B + B^T) = \lambda\varphi(A) + \mu\varphi(B)$$

Т1

Нека V и V' са ЛП (крайномерни), $\dim V = n < \infty$

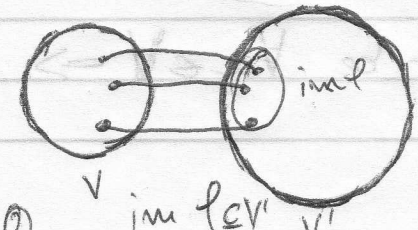
Тогава за всеки базис $e_1 - e_n$ на V и всяка система от вектори $b_1 - b_n$ на V' съществува единствено линейно изображение $\varphi : V \to V'$ т.че $\varphi(e_1) = b_1$, $\varphi(e_2) = b_2$ – $\varphi(e_n) = b_n$

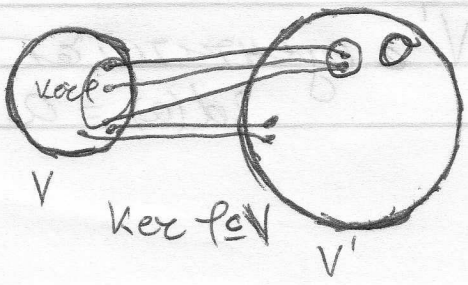
Нека V и V' са ЛП и $\varphi : V \to V'$

$\ker\varphi = \{a \mid a \in V, \varphi(a) = \mathcal{O}\}$ ядро на изображението

Всички вектори от множеството което има образ нулевият вектор

$\operatorname{im}\varphi = \{b \mid b \in V', \exists a \in V, \varphi(a) = b\}$





$\dim \operatorname{Ker} \varphi$ наричаме дефект на линейно изображение и пишем $d(\varphi) = d = \operatorname{Ker} \varphi$

$\dim \operatorname{Im} \varphi$ е ранг и бележим $r(\varphi) = r = \dim \operatorname{Im} \varphi$

$$r + d = \dim V$$

Пример

1) $\mathcal{O} : V \to V'$ нека $a \in V$, $a \in \operatorname{Ker} \mathcal{O} \iff$

$\mathcal{O}(a) = \mathcal{O}' \iff a \in V$

Тогава $\operatorname{Ker} \mathcal{O} = V$

Нека $b \in \operatorname{Im} \mathcal{O} \iff \exists a \in V, \mathcal{O}(a) = b \iff b = \mathcal{O}'$

$$\operatorname{Im} \mathcal{O} = \{\mathcal{O}'\}$$

2) $id : V \to V'$, $a \in V$

$a \in \operatorname{Ker}(id) \iff id(a) = \mathcal{O} \iff a = \mathcal{O}$, $\operatorname{Ker}(id) = \{\mathcal{O}\}$

$b \in \operatorname{Im}(id) \iff \exists a \in V, id(a) = b \iff a = b \iff b \in V$

$\operatorname{Im}(\varphi) = V$

3) Нека $V = P^4$

$f(x) \in \operatorname{Ker} \varphi \iff \delta(f) = \mathcal{O} \iff$

$f'(x) = 0x^n + 0x^{n-1} + \dots + 0 = 0 \iff f(x) = c, c \in F$

$\operatorname{Ker} \varphi = \{c \mid c \in F\}$

$g \in \operatorname{Im}(\delta) \iff \exists f(x) \in P^4, \delta(f(x)) = g(x), g(x) = f'(x)$

$g(x) = a_0 x^4 + a_1 x^3 + a_2 x^2 + a_3 x + a_4$

$f(x) = \frac{a_0}{5} x^5 + \frac{a_1}{4} x^4 + \frac{a_2}{3} x^3 + \frac{a_3}{2} x^2 + a_4 x + a_5$

искане $f(x) \in P^4 \iff$

$\deg(f) \leq 4 \iff a_0 = 0$

$g(x) \in \operatorname{Im}(\delta) \iff \deg g(x) \leq 3$

4) $V = M_2(F)$

$\varphi : V \to V$, $\varphi(A) = A + A^T$

$A \in \ker \varphi \iff \varphi(A) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \iff A + A^T = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \Rightarrow \begin{pmatrix} 2a_{11} & a_{12}+a_{21} \\ a_{21}+a_{12} & 2a_{22} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = A + A^T$

$\Rightarrow a_{11} = 0,\ a_{22} = 0,\ a_{12}+a_{21} = 0,\ a_{21}+a_{12} = 0$

антисиметрична т. е.

$A = -A^T$

$B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$ $\quad B \in \operatorname{Im} \varphi \iff \exists A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ т.ка

$\varphi(A) = B$ т. е. $A + A^T = B$

$\begin{pmatrix} 2a_{11} & a_{12}+a_{21} \\ a_{21}+a_{12} & 2a_{22} \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \Rightarrow$

$a_{22} = \frac{b_{22}}{2}$, $a_{11} = \frac{b_{11}}{2}$, $\quad \forall b_{11}, \forall b_{22}$

$\left.\begin{aligned} b_{12} &= a_{12} + a_{21} \\ b_{21} &= a_{21} + a_{12} \end{aligned}\right\} \Rightarrow b_{12} = b_{21} \Rightarrow B = B^T$

т. е. B е симетрична

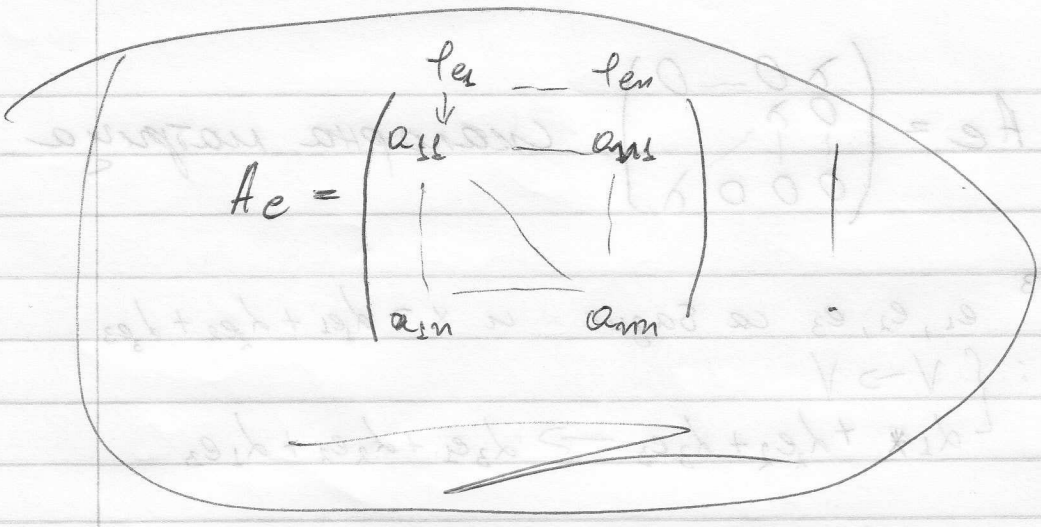
Опр

Нека $e_1 - e_n$ е базис на V и $\varphi : V \to V$ е линеен

Тогава нека $\varphi(e_1) = a_{11}e_1 + \dots + a_{1n}e_n$

$\varphi(e_2) = a_{21}e_1 + \dots + a_{2n}e_n$

$\varphi(e_n) = a_{n1}e_1 + \dots + a_{nn}e_n$ тогава



Зад 8) Кои от следните изображения са линейни

1) $a \in V$, $\varphi: \begin{cases} V \to V \\ x \to x+a \end{cases}$

φ е линейно изображение $\Leftrightarrow \forall x,y \in V$ и $\forall \lambda, \mu \in F$

$$\varphi(\lambda x + \mu y) = \lambda \varphi(x) + \mu \varphi(y) \Leftrightarrow$$
$$\lambda(x+a) + \mu(y+a) \Rightarrow a = (\lambda+\mu)a \Rightarrow$$
$$(1-\lambda-\mu)a = 0 \quad \text{т.е } (1-\lambda-\mu)=0 \text{ или } a=0$$

2) $\varphi: V \to V$, $\lambda \in F$ е фиксиран
$\varphi(x) = \lambda x$

φ е линеен оператор $\Leftrightarrow \forall x,y \in V, \forall \mu, \nu \in F$

$$\varphi(\mu x + \nu y) = \mu(\varphi(x)) + \nu(\varphi(y))$$
$$\lambda(\mu x + \nu y) = \mu(\lambda x) + \nu(\lambda y)$$

φ е линейно изображение

Нека $e_1 - e_n$ е базис на V

по дефиниция $\varphi(e_1) = \lambda e_1 + 0 e_2 \dots + 0 e_n$

$\vdots$

$\varphi(e_n) = 0 e_1 + \dots + \lambda e_n$

или $A_e = \begin{pmatrix} \lambda & 0 & \cdots & 0 \\ 0 & \lambda & & \\ \vdots & & \ddots & \\ 0 & 0 & 0 & \lambda \end{pmatrix}$ скаларна матрица

3) $V = F^3$ e_1, e_2, e_3 са базис и $x = \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3$

$$\varphi : \begin{cases} V \to V \\ \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 \to \alpha_3 e_1 + \alpha_2 e_2 + \alpha_1 e_3 \end{cases}$$

Нека $x = \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3$

$y = \beta_1 e_1 + \beta_2 e_2 + \beta_3 e_3$

$(\lambda, \beta \in F)$

$$\varphi(\lambda x + \beta y) = \varphi(\lambda\alpha_1 e_1 + \dots + \lambda\alpha_3 e_3) + \varphi(\beta\beta_1 e_1 + \dots + \beta\beta_3 e_3) =$$

$$\varphi\big(\underbrace{(\lambda\alpha_1 + \beta\beta_1)}_{\text{разменяме 1 с 3}} e_1 + \dots + \underbrace{(\lambda\alpha_3 + \beta\beta_3)}_{\text{разменяме 1 с 3}} e_3\big) =$$

$$\lambda(\alpha_3 e_1 + \alpha_2 e_2 + \alpha_1 e_3) + \beta(\beta_3 e_1 + \beta_2 e_2 + \beta_1 e_3) =$$

$$= \lambda f(x) + \beta f(y) \Rightarrow \varphi \text{ е линейно изображение}$$

$\varphi(e_1) = e_3$ т.е. $e_1 = \underbrace{1}_{\text{за } e_3} e_1 + 0 e_2 + \underbrace{0}_{\text{за } e_1} e_3 \Rightarrow$

$$A_e = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\boxed{\varphi(\alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3) = (\alpha_2 + \alpha_3 - 1) + \alpha_3 e_3 + \alpha_1 e_3}$$

Id матрица е E_n

4) $M_{2\times 2} \longrightarrow M_{2\times 2}$ $\quad \varphi(A) = A + A^t$

$E_{11}\ E_{12}\ E_{21}\ E_{22}$

$$\varphi(E_{11}) = \begin{pmatrix}1&0\\0&0\end{pmatrix} + \begin{pmatrix}1&0\\0&0\end{pmatrix} = \begin{pmatrix}2&0\\0&0\end{pmatrix} = 2E_{11}$$

$$\varphi(E_{12}) = \begin{pmatrix}0&1\\0&0\end{pmatrix} + \begin{pmatrix}0&0\\1&0\end{pmatrix} = \begin{pmatrix}0&1\\1&0\end{pmatrix} = E_{12} + E_{21}$$

$$\varphi(E_{21}) = E_{21} + E_{12}, \quad \varphi(22) = 2E_{22}$$

$$A_e = \begin{pmatrix}2&0&0&0\\0&1&1&0\\0&1&1&0\\0&0&0&2\end{pmatrix}$$

$\delta : f \to f'$

Да се намери матрица на линейно изображение δ спрямо стандартния базис

$1\ x\ x^2\ x^3$

$\delta(1) = 0$ $\qquad \delta(x^2) = 2x$

$\delta(x) = 1$ $\qquad \delta(x^3) = 3x^2$

$$A = \begin{matrix}\\1\\x\\x^2\\x^3\end{matrix}\begin{matrix}\begin{matrix}\delta(1)&&&\delta(x^3)\end{matrix}\\\begin{pmatrix}0&1&0&0\\0&0&2&0\\0&0&0&3\\0&0&0&0\end{pmatrix}\end{matrix}$$

$1 = 1.1 + 0x + _ + 0x^3$

$2x = 0.1 + 2x + _ + 0x^3$

Нека $e_1e_2e_3$ е базис на V, $\varphi \in \mathrm{Hom}\, V$, който изобразява векторите $a_1a_2a_3$ съответно във векторите $b_1b_2b_3$. Търси се матрицата на φ в дадения базис

a) $a_1 = e_1 + 2e_2 + e_3$, $a_2 = 2e_1 + e_3$, $a_3 = e_1 + e_3$

$b_1 = 4e_1 + 2e_2 + 5e_3$, $b_2 = e_1 + e_2$, $b_3 = e_3$

$f(a_1) = b_1$, $f(a_2) = b_2$, $f(a_3) = b_3$

$f(e_1) = a_{11}e_1 + a_{12}e_2 + a_{13}e_3$

$f(e_2) = a_{21}e_1 + a_{22}e_2 + a_{23}e_3$

$f(e_3) = a_{31}e_1 + a_{32}e_2 + a_{33}e_3$

$$\begin{cases} f(e_1 + 2e_2 + e_3) = 4e_1 + 2e_2 + 5e_3 \\ f(2e_1 + e_3) = e_1 + e_2 \\ f(e_1 + e_3) = e_3 \end{cases}$$

$$\begin{cases} f(e_1) + 2f(e_2) + f(e_3) = 4e_1 + 2e_2 + 5e_3 \\ 2f(e_1) + f(e_3) = e_1 + e_2 \\ f(e_1) + f(e_3) = e_3 \end{cases}$$

$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} f(e_1) \\ f(e_2) \\ f(e_3) \end{pmatrix} = \begin{pmatrix} 4e_1 & 2e_2 & 5e_3 \\ e_1 & e_2 & 0 \\ 0 & 0 & e_3 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 4e_1 & 2e_2 & 5e_3 \\ 2 & 0 & 1 & e_1 & e_2 & 0 \\ 1 & 0 & 1 & 0 & 0 & e_3 \end{array}\right) \Rightarrow - \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & e_1 & e_2 & -e_3 \\ 0 & 2 & 0 & 4e_1 & 2e_2 & 4e_3 \\ 1 & 0 & 1 & 0 & 0 & e_3 \end{array}\right) :2$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & e_1 & e_2 & -e_3 \\ 0 & 1 & 0 & 2e_1 & e_2 & 2e_3 \\ 0 & 0 & 1 & -e_1 & -e_2 & 2e_3 \end{array}\right) \Rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} f(e_1) \\ f(e_2) \\ f(e_3) \end{pmatrix} = \begin{pmatrix} e_1 & e_2 & -e_3 \\ 2e_1 & e_2 & 2e_3 \\ -e_1 & -e_2 & 2e_3 \end{pmatrix}$$

$$\left.\begin{array}{l} f(e_1) = e_1 + e_2 - e_3 \\ f(e_2) = 2e_1 + e_2 + 2e_3 \\ f(e_3) = -e_1 - e_2 + 2e_3 \end{array}\right\} \Rightarrow A_e = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 1 & 2 \\ -1 & -1 & 2 \end{pmatrix}$$

б) $a_1 = 2e_1 + 3e_2 + 5e_3$, $a_2 = e_2 + 2e_3$, $a_3 = e_1$

$b_1 = e_1 + e_2 + e_3$, $b_2 = e_1 + e_2$, $b_3 = 2e_1 + e_2$

от a от b

$$\left(\begin{array}{ccc|ccc} 2 & 3 & 5 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 & 1 & 0 \\ 1 & 0 & 0 & 2 & 1 & 2 \end{array}\right) \begin{matrix} \\ \\ \leftarrow \end{matrix} -2 \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 1 & 2 \\ 0 & 1 & 2 & 1 & 1 & 0 \\ 0 & 3 & 5 & -3 & -1 & -3 \end{array}\right) \leftarrow -3 \sim$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 1 & 2 \\ 0 & 1 & 2 & 1 & 1 & 0 \\ 0 & 0 & -1 & -6 & -4 & -3 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 1 & 2 \\ 0 & 1 & 0 & -11 & -7 & -6 \\ 0 & 0 & 1 & 6 & 4 & 3 \end{array}\right)$$

$$A_{e'}^{e} = \begin{pmatrix} 2 & -11 & 6 \\ 1 & -7 & 4 \\ 2 & -6 & 3 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & f(e_1) \\ 0 & 1 & 0 & f(e_2) \\ 0 & 0 & 1 & f(e_3) \end{array}\right) = \begin{pmatrix} 2e_1 + e_2 + 2e_3 \\ -11e_1 - 7e_2 - 6e_3 \\ 6e_1 + 4e_2 + 3e_3 \end{pmatrix}$$

$$Ae = \begin{pmatrix} 2 & -11 & 6 \\ 1 & -7 & 4 \\ 2 & -6 & 3 \end{pmatrix}$$

в) $a_1 = 2e_1 + 3e_3$, $a_2 = 4e_1 + e_2 + 5e_3$, $a_3 = 3e_1 + e_2 + 2e_3$

$b_1 = e_1 + 2e_2 - e_3$, $b_2 = 4e_1 + 5e_2 - 2e_3$, $b_3 = e_1 - e_2 + e_3$

$$\left(\begin{array}{ccc|ccc} 2 & 0 & 3 & 1 & 2 & -1 \\ 4 & 1 & 5 & 4 & 5 & -2 \\ 3 & 1 & 2 & 1 & -1 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 3 & 6 & -3 \\ 2 & 0 & 3 & 1 & 2 & -1 \\ 1 & 1 & -1 & 0 & -3 & 2 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & -4 & 2 \\ 1 & 0 & 3 & 3 & 6 & -3 \\ 1 & 1 & -1 & 0 & -3 & 2 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & -4 & 2 \\ 0 & 1 & -1 & 2 & 1 & 0 \\ 0 & 0 & 3 & 5 & 10 & -5 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & -4 & 2 \\ 0 & 1 & 0 & 11/3 & 13/3 & -5/3 \\ 0 & 0 & 1 & 5/3 & 10/3 & -5/3 \end{array}\right)$$

$$\frac{1}{3}\begin{pmatrix} -6 & 11 & 5 \\ -12 & 13 & 10 \\ 6 & -5 & -5 \end{pmatrix} = Ae$$

2) $a_1 = e_1 + 2e_2 - 3e_3$, $a_2 = 3e_1 + 2e_2 - 4e_3$, $a_3 = 2e_1 - e_2$

$b_1 = e_1 - 3e_2$, $b_2 = 10e_1 + 2e_2 + 7e_3$, $b_3 = 10e_1 + 7e_2 + 8e_3$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & -3 & 1 & -3 & 0 \\ 3 & 2 & -4 & 10 & 2 & 7 \\ 2 & -1 & 0 & 10 & 7 & 8 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 2 & -3 & 1 & -3 & 0 \\ 0 & -4 & 5 & 7 & 11 & 7 \\ 0 & -5 & 6 & 8 & 13 & 8 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 2 & -3 & 1 & -3 & 0 \\ 0 & -4 & 5 & -1 & -2 & -1 \\ 0 & -5 & 6 & 8 & 13 & 8 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & -3 & 1 & -3 & 0 \\ 0 & 1 & -1 & -1 & -2 & -1 \\ 0 & 0 & 1 & 3 & 3 & 3 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 2 & -3 & 1 & -3 & 0 \\ 0 & 1 & 0 & 2 & 1 & 2 \\ 0 & 0 & 1 & 3 & 3 & 3 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 10 & 6 & 9 \\ 0 & 1 & 0 & 2 & 1 & 2 \\ 0 & 0 & 1 & 3 & 3 & 3 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 6 & 4 & 5 \\ 0 & 1 & 0 & 2 & 1 & 2 \\ 0 & 0 & 1 & 3 & 3 & 3 \end{array}\right) \Rightarrow \left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right)\left(\begin{array}{c} f(e_1) \\ f(e_2) \\ f(e_3) \end{array}\right) = \left(\begin{array}{c} 6e_1 + 4e_2 + 5e_3 \\ 2e_1 + e_2 + 2e_3 \\ 3e_1 + 3e_2 + 3e_3 \end{array}\right) \Rightarrow$$

$$A_e = \left(\begin{array}{ccc} 6 & 2 & 3 \\ 4 & 1 & 3 \\ 5 & 2 & 3 \end{array}\right)$$